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ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
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A Programme for the Geometrical Reconstruction of Curved Tracks in a Bubble Chamber

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Introduction

The input part of the Mercury Programme for the reconstruction of curved tracks in bubble chambers is described in CERN 60- 11 "An input programme for measurements of track chamber photographs" by G.R. Macleod. There it is described how the IEP*) measurements together with their identifying labels are input and stored in the computer; also the other geometrical information necessary for the reconstruction of tracks.

Bubble chamber tracks are expected to be helices in first approximation, representing the motion of a charged particle in a quasi-uniform magnetic field perpendicular to the front glass. Measurements are taken serially on each stereoscopic view, therefore usually not corresponding to single points in space.

The present report gives a description of the sequence of calculations for the geometrical reconstruction of each measured event following the input to the computer of the necessary information by the Input Programme. It also includes a specification of the conventions and limitations imposed on the input data by the geometrical reconstruction programme and the meaning of the numbers output as results.

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*) IEP (Instrument for the Evaluation of Photographs) is the term used to designate the CERN photograph measuring instruments for which the programme has been primarily written.

Notation

It has been attempted - not always successfully - to use a consistent notation. Nearly all terms have been explained when they are first used, except those such as 'label', 'title', etc., which are already defined in CERN 60-11. Usually measurement has been used meaning an IEP measurement pair, and point meaning a point in the chamber measured with a label of type AA.

The subscript i has been used to indicate that the quantity so subscripted is taken over the whole view. The subscript (or occasionally superscript) j is used to indicate that the quantity so subscripted is taken over all the views. If both i and j occur together the quantity is taken over the whole view on every view.

Exceptions such as the use of i in the appendices should be obvious from the context. Sometimes the subscripts are dropped to make the equations less cumbersome.

Most of the symbols used are defined by figures (iv) or (iii). Important general conventions are

(xyz) refer to co-ordinates in the chamber reference system

(XY) refer to the IEP measurements on the photo (in fringes).

1. General Description

The geometrical reconstruction programme will be called the geometry programme. It may be regarded as being entirely self-contained, except that it expects to find information stored in the computer by the input programme. It could, of course, be used with any input programme which places the same information in the same places in the computer, and which has the same links with the geometry programme. These links between the input and geometry programmes are shown in Fig. (i).

The geometry programme is entered (1) when title 1 has been read in and (2) when title 3 and the measurements of one event have been read in and stored. At (1) some general calculations required for all succeeding events are performed, and control returned to the input programme to read title 2 etc. At (2) the event is reconstructed and the results or a fault number output, and control returned to the input to read in the next event.

2. Computation at the end of title 1

After title 1, the calculation of some quantities which are required for every event is performed.

2.1 The sines and cosines ($\sin \theta_{rs}$ $\cos \theta_{rs}$) of angles of lines joining cameras are found. θ_{rs} is the angle which the line joining the lenses of the cameras r, s in the xy plane makes with the x-axis.

2.2 The apparent positions (F'_{ij} G'_{ij}) on the back of the front glass of all fiducial marks, front and back, are found, by the method of appendix I. These co-ordinates serve as the reference frame for the reconstruction of each event.

3. Computation for each event

When all the measurements and labels for the event have been read in the computation proceeds as in the flow diagram of Fig. (ii). The co-ordinate system is shown in Fig. (iv). The $z = 0$ plane is the back of the front glass. z is positive towards the cameras, and therefore, of course, negative in the chamber. The optical axis of each camera lens must be perpendicular to the front glass though the cameras may be at different distances from the glass.

3.1 Coefficients of transformation to the back of the front glass

For each view in turn, the transformation coefficients α_m are found which relate the IEP measurements on that view to their apparent measurements on the back of the front glass. For this purpose the IEP measurements (F_i G_i) of the fiducial marks are substituted in the transformation equations

$$\begin{aligned} F_i' &= \alpha_1 + \alpha_2 F_i + \alpha_3 G_i \\ G_i' &= \alpha_4 + \alpha_5 F_i + \alpha_6 G_i \,, \end{aligned} \qquad (1)$$

where F_i' G_i' are the apparent fiducial mark co-ordinates found in 2.2.

If there are at least four fiducial marks measured, as the programme requires that there must be, Eqs. (1) can each be solved by least squares to yield the α_{mj} for the view j. A check is also made by back substitution of the results that the fiducial marks were measured within a set tolerance.

4. Reconstruction of points

Corresponding points, i.e. all points measured with a label of type AA on at least two views, have their co-ordinates (xyz) found.

The transformation coefficients α_{mj} found in 3.1 are used to find the apparent co-ordinates on the back of the front glass, i.e. the plane z = 0 and hence, by the method of Appendix I, the coefficients of the equation of the light-rays through the point in the chamber from the camera lenses.

We then have two equations representing such a light-ray through the point for each view :

$$x = F_x^j z + G_x^j$$
$$y = F_y^j z + G_y^j \quad . \qquad (2)$$

Measurements on two views are sufficient as they provide two pairs of equations of type (2), making four equations for the three quantities x,y,z. All measurements given for the point are used and Eqs. (2) solved by least squares to give (xyz) together with their standard errors.

If the point forms an end of a track then the event is rejected if the least squares errors are too large as the co-ordinates of an apex are used in the reconstruction of a track, and therefore they must be reasonably correct.

5. Reconstruction of tracks

The fitting of a helix to a track proceeds by the stages shown in the flow diagram (Fig. (ii)). The several stages are briefly as follows:

Check the accuracy of measurements by fitting a circle through the measurements on the photograph, find all the coefficients F_{xi}^{j}, G_{xi}^{j}, F_{yi}^{j}, G_{yi}^{j} defining the equations of the light-rays intersecting the track and corresponding to measurements, find an approximate helix using two views, find the best helix by a least squares process using all the views and, finally, find a correction to the helix to take account of the variation of curvature along the track.

5.1 Checks and preliminary calculations on photograph

To check the measurements $X_1 Y_1$ -- $X_n Y_n$ on the photograph, the track is first made nearly parallel to the X-axis by rotating through an angle $\tan^{-1}(Y_n - Y_1)/(X_n - X_1)$. Then in the new axis system $X_T Y_T$ a circle is fitted by least squares

$$Y_T = a_o + a_1 X_T + a_2 (X_T^2 + Y_T^2). \qquad (3)$$

The event is rejected if more than two points are greater than the tolerated number of fringes away from this circle. The point only is rejected if there is only one bad measurement on the track.

5.2 The coefficients in the equations

$$\begin{aligned} x &= F_{xi} z + G_{xi} \\ y &= F_{yi} z + G_{yi} \end{aligned} \qquad (4)$$

of each light ray through the track are found exactly as for the points in chapter 4, i.e. the apparent co-ordinates of a measurement as reproduced to the back of the front glass are found using the linear transformation coefficients α_m of Eqs. (1), then by the method of Appendix I, the equations of the light rays are found using the optical constants. We then have the four coefficients F_{xi}, G_{xi}, F_{yi} and G_{yi} for each measurement pair on the track; there will be at least three sets of values of these coefficients on each of at least two views of the track. These light rays will be called the "reconstruction lines". In the case of the measurement of tracks as distinct from the case of chapter 4 there are no corresponding reconstruction lines on different views so that the (xyz) of Eq. (4) cannot be found directly.

5.2.1 Once the coefficients of the reconstruction lines have been found, a check is made to see that the track has its apex point measured on at least two views, e.g. to see if an event containing the track AB has the point AA measured on at least two views. It may in fact have the point AA measured on one view only or not at all. In the former case the values of F_x, G_x, F_y, G_y given by this one measurement are used, to interpolate in the values F_{xi}, G_{xi}, F_{yi}, G_{yi} of another view by the method of Appendix II to find a near corresponding point to serve as a first approximate apex of the track. In the latter case (i.e. no measurement of AA given) the first measurement of the track on one view is used to give values of F_x, G_x, F_y, G_y for this interpolation.

5.3 First approximate helix

The next problem is to find a first approximate helix to fit the track.

5.3.1 To this end, two views are selected as being the best ones for seeing this particular track.

The first view chosen is that in which the track is viewed most nearly as an orthogonal projection, i.e. that in which the average value of $(F_x^2 + F_y^2)$ is smallest. This view will be called the pivotal view.

The second view chosen is such that the line joining the first and second camera lenses is most nearly perpendicular to the track, in the xy plane. The second view cannot be chosen until an approximate direction of the track has been found for the first time using the pivotal view. (See next section.) This second view will be called the anti-pivotal view.

5.3.2 The approximate co-ordinates of the apex of the track must now be known (having been computed in chapter 4 or 5.2.1). Let these co-ordinates be (ABC). The very first approximation to the helix is the best circle through the intersection of all the reconstruction lines from the pivotal view with the plane $z = C$. If this helix is made to pass through (ABC) it has the equation in x and y:

$$(x - A)^2 + (y - B)^2 + \lambda_1(x - A) + \lambda_2(y - B) = 0 . \qquad (5)$$

This is a circle passing through (AB) and with radius $\frac{1}{2}\sqrt{\lambda_1^{\,2} + \lambda_2^{\,2}}$. Its centre is $(\frac{1}{2}(2A - \lambda_1), \frac{1}{2}(2B - \lambda_2))$. λ_1 and λ_2 are found by least squares using the Eqs. (4) for x and y and putting for the very first approximation $z = C$. It is at this stage that the best anti-pivotal view can be found in the case where there are more than two cameras. The direction of the track is now approximately known and if the tangent at the apex makes an angle ϕ with the x-axis, and r is the number of the pivotal view, s is found such that $(\sin\phi \cos\theta_{rs} - \cos\phi \sin\theta_{rs})$ gives the largest value.

Eq. (5) can now be regarded as defining a circular cylinder with generators parallel to the z-axis. We then find where the reconstruction lines from the anti-pivotal view intersect this cylinder. We do this by substituting Eqs. (4) in Eq. (5). This gives a quadratic equation in z for each reconstruction line. The quadratic expression in $z' = (z - C)$ is easier to handle. Thus from Eqs. (4):

$$(x - A) = F_x z' + (F_x C + G_x - A) \equiv F_x z' + \phi_x$$

$$(y - B) = F_y z' + (F_y C + G_y - B) \equiv F_y z' + \phi_y .$$

This gives a quadratic in z' on substitution in Eq. (5)

$$az'^2 + bz' + c = 0 , \qquad (6)$$

where

$$a \equiv F_x^2 + F_y^2$$

$$b \equiv (2\phi_x F_x + 2\phi_y F_y + \lambda_1 F_x + \lambda_2 F_y)$$

$$c \equiv (\phi_x^2 + \phi_y^2 + \lambda_1 \phi_x + \lambda_2 \phi_y)$$

$$z' = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (6a)$$

where the lower numerical value is always taken and has always been found to be satisfactory.

The helix which is being fitted to the data is

$$x' = \rho(\text{Cos}\ \theta - 1) \qquad (7a)$$

$$y' = \rho\ \text{Sin}\ \theta \qquad (7b)$$

$$z' = \rho\ \theta\ \tan\alpha \qquad (7c)$$

where the $(x'y'z')$ axis system is the original (xyz) axis system rotated through an angle β about the z-axis and moved to a new origin (ABC), 'ρ' is the radius of the cylinder on which the helix lies and α is the dip angle (see Fig. iv).

The first approximation to the helix so far has been to put

$$\tan\alpha = 0$$

$$\rho = \tfrac{1}{2}\sqrt{\lambda_1^2 + \lambda_2^2}$$

$$\beta = \tan^{-1}\left(\frac{\lambda_2}{\lambda_1}\right) , \qquad (8)$$

where λ_1 and λ_2 have been found from Eq. (5). The third of Eqs. (8) comes from the fact that β is the angle which the radius to the curve at the apex (ABC) makes with the z-axis. It is thus given by

$$\beta = \tan^{-1} \left\{ \frac{B - \frac{1}{2}(2B - \lambda_2)}{A - \frac{1}{2}(2A - \lambda_1)} \right\} = \tan^{-1} \left(\frac{\lambda_2}{\lambda_1} \right). \qquad (9)$$

Similarly

$$(\beta + \theta) = \tan^{-1} \frac{y_i - \frac{1}{2}(2B - \lambda_2)}{x_i - \frac{1}{2}(2A - \lambda_1)}, \qquad (9a)$$

where $(x_i\ y_i\ z_i)$ is another point on the helix with parameter θ_i in Eqs. (7).

Thus, for each measurement on the anti-pivotal view we can find z_i' using Eq. (6), we can then find x_i and y_i using Eqs. (4) and hence θ_i from Eqs. (9 and 9a). From Eq. (7c) we then have

$$\tan \alpha = \left(\frac{z_i'}{\rho\theta_i} \right).$$

Tan α should be the same for every measurement on the anti-pivotal view, but of course it is not exactly and the average is taken.

The next approximation to the helix is to take this average value of tan α in Eq. (7c). New values for ρ and β have to be found. It is required then to find better values of x_i and y_i to substitute in Eqs. (5).

Returning then to the pivotal view; for each measurement in turn the original (x_i, y_i) can be used in Eqs. (9 and 9a) to find θ_i. This θ_i is then substituted in Eq. (7c) to give a new value of (x_i, y_i). Using these new values of (x,y) the Eqs. (5) are solved to give the new values of λ_1, and λ_2, which are substituted in Eqs. (9 and 9a) etc.

The iteration is then obvious, it continues until two successive values of tan α do not differ by more than .001. The convergence is somewhat slow and the programme has a limit of ten iterations. If it has not converged by then the mean of the last iterations is taken, as it has been found that the process sometimes shows a slow oscillatory convergence.

It should be noted here that as well as finding the coefficients in an approximate helix, we also need for the next process a set of values of θ_{ij} the parameter in the helix, one for each measurement i on every view j. So far we only have θ_{ij} for the pivotal and anti-pivotal views. For the other view(s) it is necessary to make some other approximation. In fact what is done is to take the corresponding value of θ_i on the pivotal view for this other view and make that do. It was originally intended to find the θ values for all other views as has been described for the anti-pivotal view. But in many cases this gives imaginary roots in solving Eq. (6).

5.3.3 If this iteration procedure does lead to imaginary roots in Eq. (6) anyway, or diverges, the points on one track image are established by an approximate "near corresponding points" method. This is described in Appendix II.

5.3.4 Numerical difficulties

There are two places where cancellation may cause severe loss of accuracy in the iteration to find the first approximate helix, if it is not taken care of.

5.3.4.1 In Eqs. (9 and 9a) if θ is very small relative to β, it may not be found very accurately using these equations. If θ_i is very small

$$\tan \theta_i \simeq \left(\frac{\sqrt{(x_i - A)^2 + (y_i - B)^2}}{\rho} \right) \qquad (9b)$$

and the value of this expression is used as the criterion for which equations to use. If the R.H.S. of Eq. (9b) is less than .05, then Eq. (9b) is used to evaluate (θ).

5.3.4.2 In Eq. (6a) if '4a c' is very small relative to 'b^2' cancellation may occur. A check is made and if $\left(\frac{ac}{b^2/4}\right) < .0001$ then

$$z' = \frac{1}{a}\left\{-\frac{b}{2} \pm \frac{b}{2}\sqrt{1 - \left(\frac{ac}{b^2/4}\right)}\right\}$$

$$\cong \quad -\frac{1}{a}\left\{\frac{b}{2} - \frac{1}{2}\left(\frac{ac}{b^2/4}\right) + \frac{1}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{ac}{b^2/4}\right)^2\right\}$$

and this approximation is taken.

5.3.5 Under some circumstances the programme will treat a track as straight. This means that the track is ascribed a fixed large radius - in fact, ten times the number "maximum measurable radius" ρ_{max} in title 1 - and the computation will be somewhat different.

If the ρ_{max} is negative, all tracks will be treated as "straight", i.e. with ($\rho_{max} \times 10$) as radius.

In 5.3.2 if $\rho = \frac{1}{2}\sqrt{\lambda_1^2 + \lambda^2}$ is greater than the ρ_{max}, the radius will also subsequently be fixed for this particular track at $10 \times \rho_{max}$. If the label of the track has a letter beyond the ℓ^{th} in the alphabet, where 'ℓ' is a number in title 1 miscellaneous constants (see Chapter 6), the radius ρ of this track will also be fixed at $10 \times \rho_{max}$.

For straight tracks the same programme is used but slightly different courses are taken at some stages. In particular

5.3.5.1 Eq. (5) reduces to

$$\left(\frac{\lambda_2}{\lambda_1}\right) = -\left(\frac{\Sigma(x_i - A)}{\Sigma(y_i - B)}\right).$$

5.3.5.2 The quadratic Eqs. (6) reduce to

$$z' = -\left(\frac{\lambda_1\phi_x + \lambda_2\phi_y}{\lambda_1 F_x + \lambda_2 F_y}\right)$$

5.3.5.3 The difficulties of cancellation mentioned in 5.3.4 apply especially to the case of 'straight' tracks.

5.4 Final least squares fit of helix to track

There are now

a) for the helix, approximate values of ρ, tan α, β as well as the apex co-ordinates (ABC)

b) for each measurement on the track, a value of each of $F_{xi}{}^j$, $G_{xi}{}^j$, $F_{yi}{}^j$, $G_{yi}{}^j$, of Eqs. (4) and an approximate value of θ_{ij} the parameter in the helix (Eqs. (7)) where the corresponding reconstruction line intersects the helix.

5.4.1 The problem is to find small corrections to the coefficients to the helix so that the $(x_i y_i z_i)$ satisfy simultaneously Eqs.(4) and at the same time Eqs. (7). Bearing in mind the definition of the $(x'y'z')$ axis system

$$x = x' \cos\beta - y' \sin\beta + A \quad (10a)$$

$$y = y' \cos\beta + x' \sin\beta + B \quad (10b)$$

$$z = z' + C \quad (10c)$$

Substituting Eqs. (10) in Eq. (4) gives two simultaneous equations in θ

$$f_1(\theta_{ij}) \equiv A + \rho(\cos\theta_{ij} - 1)\cos\beta - \rho \sin\theta_{ij} \sin\beta - F_{xi}{}^j(\rho\theta_{ij}\tan\alpha + C) - G_x = 0 \quad (11a)$$

$$f_2(\theta_{ij}) \equiv B + \rho \operatorname{Sin} \theta_{ij} \operatorname{Cos} B - \rho(\operatorname{Cos} \theta_{ij} - 1)\operatorname{Sin} \beta$$
$$- F_{yi}{}^{j}(\rho\theta_{ij} \tan \alpha + C) - G_y = 0 \;. \qquad (11b)$$

Eqs. (11a) will not in general be satisfied exactly for this value of θ_{ij}. The respective Newton corrections for the two equations are

$$(\Delta\theta_{ij})_1 = -\frac{f_1(\theta_{ij})}{f_1'(\theta_{ij})} \quad \text{and} \quad (\Delta\theta_{ij})_2 = -\frac{f_2(\theta_{ij})}{f_2'(\theta_{ij})}$$

where the dashes indicate differentiation w.r.t. θ. But Eq. (11a) and Eq. (11b) must be satisfied simultaneously for the same value of θ_{ij}. Hence

$$(\Delta\theta_{ij})_1 = (\Delta\theta_{ij})_2 \quad \text{or} \quad -\frac{f_1(\theta_{ij})}{f_1'(\theta_{ij})} = -\frac{f_2(\theta_{ij})}{f_2'(\theta_{ij})} \;. \qquad (12)$$

Eqs. (12) may be written

$$\left(\frac{f_1(\theta_{ij})}{f_1'(\theta_{ij})} - \frac{f_2(\theta_{ij})}{f_2'(\theta_{ij})}\right) \equiv f_{ij}(\rho, \tan \alpha, \beta, A, B, C) = 0 \;. \qquad (13)$$

Eq. (13) may be expanded in terms of the small corrections to the coefficients of the helix, to give the linear set of equations.

$$f_{ij}(\bar{\rho}, \overline{\tan \alpha}, \bar{\beta}, \bar{A}, \bar{B}, \bar{C})$$
$$+ \frac{\partial f_{ij}}{\partial \rho} \Delta\rho + \frac{\partial f_{ij}}{\partial \tan \alpha} \Delta \tan \alpha + \frac{\partial f_{ij}}{\partial \beta} \Delta\beta + \frac{\partial f_{ij}}{\partial B} \Delta B + \frac{\partial f_{ij}}{\partial C} \Delta C = 0 \qquad (14)$$

where the barred quantities are the approximate coefficients of the helix found in section 5.3. A is not found as only two of (ABC) may be found. There is an equation of type (Eq. (14)) for every measurement on the track on every view. These equations are solved by least squares for the five small quantities Δq_r and the standard errors and covariances on those quantities.

The coefficients in the Eq. (14) are computed as follows: $f_1(\theta_{ij})/\rho$ and $f_2(\theta_{ij})/\rho$ are computed as defined by Eqs. (11) divided by ρ. Then

$$f_1'(\theta_{ij})/\rho = - \operatorname{Sin}\theta_{ij} \operatorname{Cos}\beta - \operatorname{Cos}\theta_{ij} \operatorname{Sin}\beta - F_{xi}^{j} \tan\alpha \qquad (15a)$$

and

$$f_2'(\theta_{ij})/\rho = \operatorname{Cos}\theta_{ij} \operatorname{Cos}\beta - \operatorname{Sin}\theta_{ij} \operatorname{Sin}\beta - F_{yi}^{j} \tan\alpha \qquad (15b)$$

Whence the various coefficients are (dropping the i and j for convenience):

$$\rho \cdot \frac{\partial f}{\partial \rho} = \left[f_1(\theta)/\rho + \frac{1}{\rho}\{F_x C + G_x - A\}\right] / (f_1'(\theta)/\rho) - \left[f_2(\theta)/\rho + \frac{1}{\rho}\{F_y C + G_y - B\}\right] / (f_2'(\theta)/\rho) \qquad (16a)$$

$$\frac{\partial f}{\partial \tan\alpha} = \left[-F_x \theta\right] / (f_1'(\theta)/\rho) - \left[-F_y \theta\right] / (f_2'(\theta)/\rho) \qquad (16b)$$

$$\frac{\partial f}{\partial \beta} = \left[- \operatorname{Sin}\beta(\operatorname{Cos}\theta - 1) - \operatorname{Sin}\theta \operatorname{Cos}\beta\right] / (f_1'(\theta)/\rho) - \left[- \operatorname{Sin}\beta \operatorname{Sin}\theta + (\operatorname{Cos}\theta - 1) \operatorname{Cos}\beta\right] / (f_2'(\theta)/\rho) \qquad (16c)$$

$$\rho \cdot \frac{\partial f}{\partial B} = - (1/(f_2'(\theta)/\rho) \qquad (16d)$$

$$\rho \cdot \frac{\partial f}{\partial C} = \left[- F_x \right] / (f_1'(\theta)/\rho) - \left[- F_y \right] / (f_2'(\theta)/\rho) \quad (16e)$$

The R.H.S. is

$$\left[f_2(\theta)/\rho \right] / (f_2'(\theta)/\rho) - \left[f_1(\theta)/\rho \right] / (f_1'(\theta)/\rho) \, . \quad (16f)$$

If the track lies nearly parallel to the y-axis at the apex (in fact if $(\beta) < .075$) then ΔA is found instead of ΔB. Instead of Eq. (16d) there is

$$\rho \cdot \left(\frac{\partial f}{\partial A} \right) = 1/(f_1'(\theta)/\rho) \, .$$

Before evaluating the coefficients of the Eq. (14) it is better at once to find an improved value of θ_{ij} to substitute in the Eqs. (16) especially since nearly all the necessary programme is there anyway. In fact the first time that the quantities $f_1(\theta)$, $f_2(\theta)$, $f_1'(\theta)$, $f_2'(\theta)$ are evaluated

$$\theta_{ij} \text{ is put } = \bar{\theta}_{ij} + \tfrac{1}{2} \left(- \frac{f_1(\bar{\theta}_{ij})}{f_1'(\bar{\theta}_{ij})} - \frac{f_2(\bar{\theta}_{ij})}{f_2'(\bar{\theta}_{ij})} \right)$$

i.e. adding the mean of the Newton Corrections to θ, for the two equations (11). $f_1(\theta)$, $f_2(\theta)$, $f_1'(\theta)$, $f_2'(\theta)$ are re-evaluated using the new value of θ_{ij} before evaluating the coefficients of Eqs. (16).

The procedure is repeated with the new values of the coefficients of the helix until the Δ's are sufficiently small. In practice it is difficult to find a good criterion for the sufficiency of the convergence, and an upper limit is placed on the number of iterations to save computer time.

5.4.2 This is the general method for the final least squares fit of the helix. There are several possible reasons why the techniques may be varied slightly.

5.4.2.1 In the case of a straight track the radius will already have been fixed at several metres (see section 5.3.5). In this case the procedure is exactly the same except that $\Delta\rho$ is not found, nor, of course, the errors and covariances associated with it.

5.4.2.2 If the radius ρ becomes negative, or if $\Sigma\Delta q_r^2$ starts to increase, or if the value of the co-ordinate C of the apex strays too far from the original, then the apex (ABC) is fixed and the Eqs. (14) are solved for $\Delta\rho$, $\Delta \tan\alpha$, $\Delta\beta$ only.

If something still goes wrong the number of iterations is limited to one so that at least the errors resulting from the least squares minimization may be available.

5.4.2.3 It is also possible that $f_1'(\theta_{ij})$ or $f_2'(\theta_{ij})$ may become very small for some value or values of θ_{ij} along the track. This means physically that the tangent to the track at this point is practically parallel to one of the co-ordinate axes.

To avoid the numerical difficulty, a check is made and if $f_1'(\theta)$ or $f_2'(\theta) < .05$, Eqs. (11) are replaced by

$$F_1(\theta) \equiv f_1'(\theta) + f_2'(\theta) = 0 \qquad (17a)$$

$$F_2(\theta) \equiv f_1'(\theta) - f_2'(\theta) = 0 \qquad (17b)$$

effectively rotating the (xy) axis through 45°. It can be shown that the Eqs. (16) remain the same with

$$1/(f_1'(\theta)/\rho) \quad \text{replaced by} \quad \left\{\frac{2f_2'(\theta)\rho}{(f_2'^2(\theta) - f_1'^2(\theta))}\right\}$$

and

$$1/(f_2'(\theta)/\rho) \quad \text{replaced by} \quad \left\{\frac{2f_1'(\theta)\rho}{(f_2'^2(\theta) - f_1'^2(\theta))}\right\}.$$

This is of course equivalent simply to multiplying the Eqs. (14) by a weighting factor but it avoids giving too great a weight to such an equation and also the inaccuracy caused by cancellation.

5.4.3 The reason for dividing the f(θ) by 'ρ' was that it was originally intended to use $\Delta(\frac{1}{\rho})$ instead of $\Delta\rho$ as the variable in Eq. (14) and anyway it keeps the numbers small. It was found, however, that this led to the null solution $\theta = 0$, $\frac{1}{\rho} = 0$. This can most easily be seen from Eqs. (11). If either of these equations is divided by 'ρ' there are terms only in $(\frac{1}{\rho})$ or $(\cos\theta - 1)$ or $\sin\theta$ so that $\theta = 0$, $\frac{1}{\rho} = 0$, satisfy the equations. This is a pity since $\Delta(\frac{1}{\rho})$ is preferable as a variable to $\Delta\rho$ because the errors on $\Delta(\frac{1}{\rho})$ are more likely to be symmetrical.

5.4.4 This leads to a discussion of the accuracy of the results of the helix fit and the validity of the standard errors and covariances. The results of an analysis of the results of the reconstruction of an artificial event are given in Appendix IV. The same event was artificially created sixty times with different random errors on the "measurements" each time and reconstructed by the geometry programme. The analysis of the results shows that in this case, at any rate, the results of the geometry programme are reliable and the standard errors are meaningful.

There seems little reason to doubt that the technique gives the best helix fit possible to the measurements and that it is worthwhile making the least squares fit using all the measurements on all the views. It is interesting to see what form the Eq. (13) takes in the extreme case of a 'bad' view namely a reconstruction line from the camera being a tangent to the track in the xy plane.

Thus

$$\frac{F_y}{F_x} = -\cot(\theta + \beta) \qquad (18a)$$

and Eq. (13) becomes, on multiplying through by

$$-\ (\mathrm{Sin}(\Theta + \beta)\ \mathrm{Cos}(\Theta + \beta)),$$

and after some reduction

$$F_y\left[A + \rho\,(\mathrm{Cos}\,\Theta - 1)\,\mathrm{Cos}\,\beta - \rho\,\mathrm{Sin}\,\Theta\,\mathrm{Sin}\,\beta\right]$$
$$-\ F_x\left[B + \rho\,\mathrm{Sin}\,\Theta\,\mathrm{Cos}\,\beta + \rho\,(\mathrm{Cos}\,\Theta - 1)\,\mathrm{Sin}\,\beta\right]$$
$$= (F_y G_x - F_x G_y)\ . \qquad (18b)$$

Eq. (18b) is independent of tan α, and C and this is what we should expect. There is no reason to suppose that this equation will have any tendency to produce an unwanted solution; there is then no harm in including this 'bad' measurement even in this extreme case.

However, the coefficients of tan α and C in Eq. (14) are now zero or very small and this will undoubtedly lead to an under**estimate** of the variance

$$\frac{\Sigma\,(Y_i - y_i)^2}{N - n}$$

of Appendix III on which the errors are based, and this will lead to an underestimate of the errors in some cases.

5.5 When all the tracks have the best helices fitted to them the tracks are arranged in alphabetic - numeric order and the results output, as explained in chapter 7.

5.6 The paragraph entitled "Variation of track from a helix" in the first version of this report has been omitted here. The general higher order curve fit in space was not successful. The least squares fitting technique has, however, been worked out for a particle of known mass in a magnetic field which varies in the chamber, (DD/IEP/61/7). Though this seems to work it has not been used at CERN.

6. Preparation of Data

Details of the labelling system for events and the punching of data for titles are given in CERN 60 - 11. The geometry programme requires other rules, which are not mentioned there, to be observed.

6.1 Titles 1 and 2

In title 1 the list of miscellaneous data at the end is now different. A typical title 1 would have the form shown on page 29. If title 2 is not used in the curved tracks geometry programme a dummy one only should be provided i.e. with opening and terminating characters but nothing in between.

6.2 Punching of data by the IEP

The rules governing the punching of events as far as they can be given are set out below. A point means a single point in the bubble chamber measured with a label of type AA. A measurement means the XY co-ordinate pair punched automatically by the IEP.

At least four fiducial marks must be measured with each event.

An event may consist of no tracks but must have at least one point.

An isolated point, i.e. one not on any track, must, of course, be measured on at least two views.

A track may be of the type A1 or AB but not 12, and must have measurements on at least two views.

The maximum angle through which a track can be measured is 180 degrees. To be safe it should be less than about 150 degrees.

A track A1 or AB may be measured without the apex AA being measured at all separately, or AA may be measured on one view only.

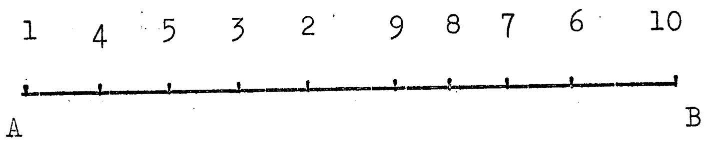
A second point on a track like BB on the track AB cannot be measured on one view only like the apex A. It need not be measured; but if it is measured it must be measured on at least two views.

If an event consists entirely of tracks the first requirement above, namely that an event to be computed must have at least one point, may be met <u>either</u> by measuring an apex on one view only, i.e. choosing one point at the beginning of a track on one view and calling that an apex AA <u>or</u> by choosing some isolated point, e.g. a fiducial mark, and measuring that on two views (or more).

If the number "maximum measurable radius" in title 1 is negative all tracks will be treated as straight, i.e. as having this fixed large radius. In any case, if this number is positive any track having a label containing a letter whose position in the alphabet exceeds or equals the number 'l' title 1 will be similarly treated. For example, if l= 22 any track which includes V,W,X,Y or Z in its label, e.g. MW or WA will be treated as straight. This facility is provided for tracks which are too short for their radii to be determinable. Also, if in the first approximation a track's radius exceeds the "maximum measurable radius", it is henceforth treated as straight in the programme.

In measuring along a track the extreme points must be the first and last measurements but in between any order may be followed. Thus the track AB may be measured in the order shown.

No. of measurement



Fiducial marks are of two kinds - front and back glass.
Front glass fiducial marks are labelled 11,22,33,44,66
Back glass fiducial marks are labelled 77,88,99,00
The number in the label refers to the order in which they are given in title 1. Thus the first front-glass fiducial mark given in title 1 has the label 11, and the first back-glass fiducial mark the label 77 etc. They may, of course, be <u>measured</u> in any order, provided they are labelled correctly.

The following table gives the limitations to be observed when using the programme.

	Min. No.	Max. No.
Cameras	2	4
Fiducial marks	4	6 front & 4 back
Points	1	20
Tracks	0	24 (1)
Measurements on each track/view	3	47 (2)
Measurements on each track on all views	6	96 (2)

NOTES : 1) Although theoretically 24 tracks are admissible it is quite likely that Number Store will be exceeded especially if there are many measurements on each. An event has been computed with 18 tracks on 3 views. Number Store in this case was practically filled: thus it is felt that this is about the practical limit, i.e. the equivalent of about 18 tracks on 3 views.

2) If the total number of measurements is greater than this, the programme reduces the number, removing measurements from the last view(s).

Some examples of short events are given below. * denotes IEP measurement. The other symbols have the meaning given to them in CERN 60 - 11

Examples of events possible

```
3' 100001,  " +1 * 11 * 22 *   AA * 33 * 44 *
              +3 * 11 * 22 *   AA * 33 * 44 * "

3' 100002,  " +2 * 11 * 22 *   44 * 33 * AA * A1 *****
              +3 * 11 * 22 *   33 * 44 * A1 ****** "
```

```
3' 100003, " +1 * 11 * 22 *   UU * A1 ******** 33 * 44 *
             +2 * 11 * 22 *        A1 *******  33 * 44 *
             +3 * 11 * 22 *   UU * A1 ******** 33 * 44 * "

3' 100004, " +1 * 11 * 22 *   AA * AB *****    33 * 44 *
             +2 * 11 * 22 *   AA * AB ******   33 * 44 *
             +3 * 11 * 22 *   AA * AB ******** 33 * 44 * "

3' 100005, " +1 * 11 * 22 * AA * BB * AB ***** 33 * 44 *
             +2 * 11 * 22 *   AA * AB ******   33 * 44 *
             +3 * 11 * 22 *   AA * BB * AB ***** 33 * 44 * "
```

7. <u>Form of the Output</u>

EVENT 123456 REF. 1002 n_1 n_2

POINT APEX STOP	A	X	ΔX	Y	ΔY	Z	ΔZ

for all points.

(i)	TRACK A1		n_3		n_4	
(ii)	ρ	tan α	β	sin β	cos β	
(iii)	$\Delta\rho$	Δtan α	$\Delta\beta$			These numbers are in floating point form.
(iv)	C_{12}	C_{13}	C_{23}			
(v)	—	—	—	—	θ_{max}	
(vi)	A	ΔA	B	ΔB	C	ΔC

for all tracks in alphabetic-numeric order A1 ...... ZY

(If the track is of type AB then the track BA will also be given with a slightly different weighting factor - heavier now at the end B, unless BB is not measured).

All lengths are in mm or cm depending on the units employed in title 1. Angles are in radians.

7.1 Meaning of numbers in the output

7.1.1 For the whole event

Event and ref. no. are self explanatory. n_1 is the number of points, n_2 is the number of tracks. There is a short length of blank tape (six figure shift characters) after each event toseparate events on the tape.

If the event fails during the geometry programme, no results will be printed but "FAULT 13 (A1,2)" for example. The explanation of the fault numbers is on page 28.

7.1.2 For each point

A POINT means an isolated point, i.e. not occurring as the end of any track.

An APEX means a point which occurs at the end of at least two tracks.

A STOP means a point which appears at the end of only one track.

(XYZ) are the co-ordinates in the chamber and ($\Delta X \Delta Y \Delta Z$) their errors. If an APEX or STOP has not been measured separately it does not occur here. If an APEX or STOP at the beginning of a track has been measured on one view only it occurs here, but all the co-ordinates and errors are zero.

7.1.3 For each track

n_3 and n_4 are small integers showing the number of iterations in different loops.

ρ, tan α, β given in line (ii) together with A, ΔA, B, ΔB, C, ΔC the co-ordinates of the "apex" and errors given in line (vi) define the helical track (see Fig. (iv)).

The helix is of the form

$$\left.\begin{aligned} x' &= \rho(\text{Cos } \theta - 1) \\ y' &= \rho \sin \theta \\ z' &= \rho \theta \tan \alpha \end{aligned}\right\} \qquad \text{(i)}$$

where the $(x'y'z')$ axis system is the original (xyz) axis system rotated through an angle β about the z axis and moved to a new origin (ABC).

In simpler terms, ρ is the radius projected on the XY plane, $\pm\ \alpha$ is the dip angle and $\beta \pm \frac{\pi}{2}$ the azimuth, where the sign ambiguity is resolved by consideration of the direction of curvature given by the sign of θ_{max}, the last number in line (v). The number θ_{max} gives the value of θ corresponding to the measurement furthest from the apex; it is positive if the rotation is anti-clockwise away from the apex, negative if clockwise in the usual convention.

If θ_{max} +ve ; dip angle $\alpha' = +\ \alpha$, azimuth $\phi = \beta + \frac{\pi}{2}$

If θ_{max} -ve ; dip angle $\alpha' = -\ \alpha$, azimuth $\phi = \beta - \frac{\pi}{2}$

Thus the direction cosines of the tangent at the apex with positive sense away from the apex are given by

$$\begin{aligned} \ell &= \text{Cos } \phi \cdot \text{Cos } \alpha' \\ m &= \text{Sin } \phi \cdot \text{Cos } \alpha' \\ n &= \text{Sin } \alpha' \text{ .} \end{aligned}$$

The other lines of the results are

line (iii) The errors on ρ, tan α, β

line (iv) The covariances $C_{(\rho,\ \tan\alpha)}$ $C_{(\rho,\beta)}$ $C_{(\tan\alpha,\ \beta)}$

line (v) The last number is the maximum value of Θ_i found on any view.

line (vi) It has already been explained that these six numbers are the co-ordinates of the "apex" and their errors. The usual technique in the programme is to find the value of the x co-ordinate and find only the y and the z co-ordinate by an iterative least squares process along with ρ, tan α, β. This can be seen in that $\Delta A = 0$, but ΔB and $\Delta C \neq 0$. However, if the track is nearly parallel to the y-axis at the apex then the y co-ordinate is fixed. If this happens $\Delta B = 0$, but ΔA and $\Delta C \neq 0$. If the iterative process used does not converge (ABC) are fixed and the computer tries again finding only ρ, tan α, β. If this happens $\Delta A = \Delta B = \Delta C = 0$. If, in addition, $n_4 = 0$ it means that it has not even then converged and only one iteration has been made. The results in this case are suspect!

The co-ordinates (BC) (or AC) found on the track, should, of course, not differ greatly from the corresponding co-ordinates given for that apex at the beginning of the results. If the apex is not measured separately, of course, the values (ABC) form the only estimate of its co-ordinates.

7.2 Different form of the output for straight tracks

The differences are as follows

line (ii) $\rho = -(\rho_{max} \times 10)$

line (iii) $\Delta\rho = 0$

line (iv) $C_{(\rho \tan \alpha)} = C_{(\rho\beta)} = 0$

line (v) All the numbers are zero or nearly so.

GEOMETRY PROGRAMME FAULTS

FAULT NUMBER	MEANING	VARIABLES
1	Not enough fiducial mark measurements	(View No.)
2	Error on fiducial mark measurements	(View No.)
3	Not enough measurements for point	(Point)
4	Point measurements but no fiducial marks given	(View No.)
5	No camera co-ordinates given	(View No.)
6	More than 47 points on line on one view *	(Line, view No. of pts.)
7	More than 96 points on line on all views *	(Line, No. of pts.)
8	Not enough views for line	(Line)
9	Not enough measurements on line per view	(Line, view)
10	Bad measurement on apex on one view	(Point, view)
11	More than one bad measurement on line per view	(Line, view)
12	Not enough good pts. left on line after 1 bad measurement.	(Line, view)
13	First or last measurement on line not at end	(Line, view)
14	First approx. gives trouble even after corr. pts.	(Line)
15	Line measurements given but no fiducial marks	(View)
16	Cannot find any approx. corr. pts.	(Line)
17	Error on apex too large	(Point)
18	Too many or too few points	(No. of points)
19	Too many tracks	(No. of tracks)

* When this fault number is printed the programme continues after reducing the number of measurements.

A typical title 1 and 2

Title 1

1'	Warning characters
12015,	Reference Number
1,	Number of media (one front glass)
+1.093,	Liquid refractive index
+1.517,	Glass refractive index
+80.0,	Glass thickness
3,	Number of cameras
+67.9, -152.2, +77.3,	x co-ordinates of cameras 1, 2, 3
+128.6, -9.5, -131.1,	y co-ordinates of cameras 1, 2, 3
+1128.0, +1128.0, +1128.0,	z co-ordinates of cameras 1, 2, 3
4,	Number of front fiducial marks
+100.8, +0.1, -99.2, -45.1,	x co-ordinates of front marks 1,2,3,4
+0.68, -99.87, +0.53, -40.2,	y co-ordinates of front marks 1,2,3,4
0,	Number of back fiducials (zero, therefore no co-ordinates follow)
5,	Number of constants
+2.0,	Allowable tolerance on fiducial mark measurements
+50.0,	Allowable tolerance on photograph measurements (in digitiser fringes)
+20.0,	Permissible error sum ($\Delta x + \Delta y + \Delta z$) on an apex
+100000.0,	Maximum calculable radius of curvature (ρ_{max})
+22,	No. of letter in alphabet (ℓ) Tracks having a letter in their label beyond the ℓ'th are computed as straight, e.g. in this case AV.
"	Terminating character

Title 2

2'	Warning characters
"	Terminating character

Co-ordinates may be in cms or mms but must be consistent.

APPENDIX I

Derivation of reconstruction line where there are several thicknesses of materials of different refractive index.

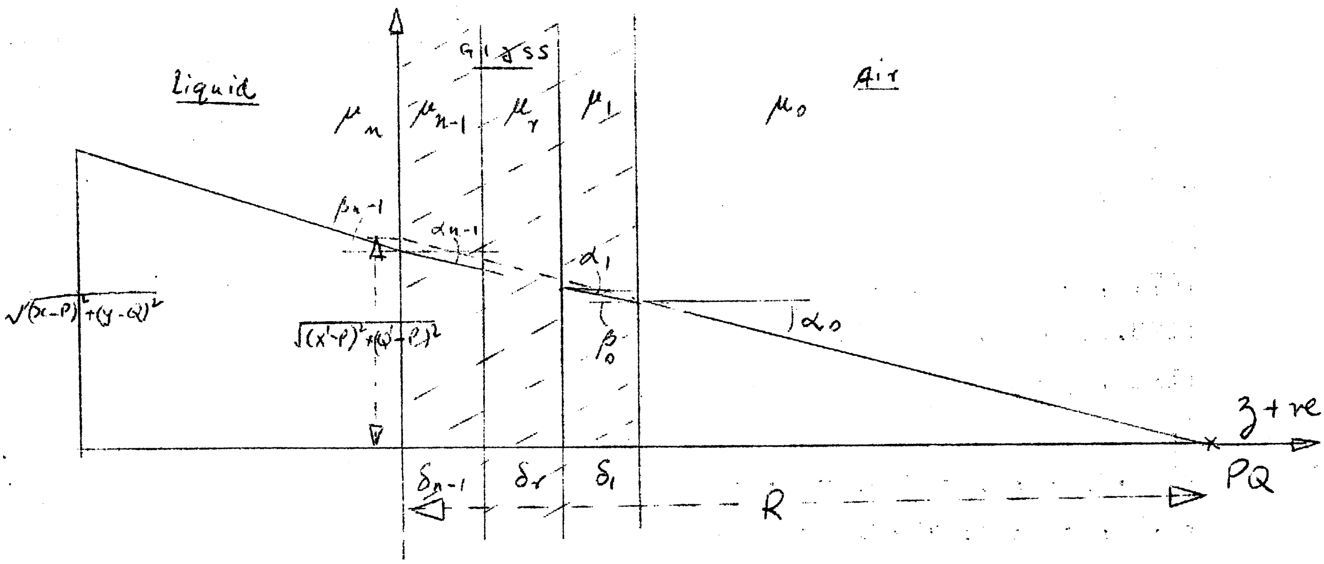


Fig. (iii) Section through a light ray and optical axis.

X′Y′ are the co-ordinates measured as apparently reproduced to the back of the front glass, (PQR) the co-ordinates of the camera lens

$$\sqrt{(x-P)^2 + (y-Q)^2} = \sqrt{(X'-P)^2 + (Y'-Q)^2} + \sum_1^{n-1} \delta_i \tan \beta_{(i-1)}$$

$$-\sum_i \delta \tan \alpha_o - z \tan \beta_{(n-1)}$$

$$\text{Now } \tan \alpha_o = \frac{\sqrt{(X'-P)^2 + (Y'-Q)^2}}{R}$$

$$\text{and } \tan \beta_i = \frac{\text{Sin}\beta_i}{\sqrt{1-\text{Sin}^2 \beta_i}} = \frac{\frac{\mu_i}{\mu_{i+1}} \text{Sin } \alpha_i}{\sqrt{1-\left(\frac{\mu_i}{\mu_{i+1}}\right)^2 \text{Sin}^2 \alpha_i}} \qquad \text{(I.1)}$$

This gives a computational sequence since $\alpha_i = \beta_{(i-1)}$, and

$$\tan \alpha_o = \frac{\sqrt{(X'-P)^2 + (Y'-Q)^2}}{R} \qquad (I.2)$$

$\therefore$ the equation of the reconstruction line becomes, (since

$$\frac{(x-P)}{\sqrt{(x-P)^2 + (y-Q)^2}} = \frac{(X'-P)}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \text{),}$$

$$x = \frac{P-X'}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \tan \beta_{(n-1)} z + \left(\frac{P-X'}{R}\right) \Sigma \delta_i$$

$$+ \Sigma_1^{n-1} \delta_i \tan \beta_{(i-1)} \left\{ \frac{X-P}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \right\} + X_1 \qquad (I.3)$$

and similarly

$$y = \frac{Q-Y'}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \tan \beta_{n-1} z + \left(\frac{Q-Y'}{R}\right) \Sigma \delta_i$$

$$+ \Sigma_1^{n-1} \delta_i \tan \beta_{(i-1)} \left\{ \frac{Y^1-Q}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \right\} + Y_1 \qquad (I.4)$$

and the iteration is as above - equations (I.1) and (I.2).

We may write (I.3) and (I.4) as

$$x = (P-X')\xi z + (P-X')\eta + X'$$

$$y = (Q-Y')\xi z + (Q-Y')\eta + Y'$$

where $\xi = \dfrac{\tan \beta_{(n-1)}}{\sqrt{(X'-P)^2 + (Y'-Q)^2}}$, $\eta = \left\{ \dfrac{\Sigma \delta i}{R} - \dfrac{\Sigma_1^{n-1} \delta_i \tan \beta_{(i-1)}}{\sqrt{(X'-P)^2 + (Y'-Q)^2}} \right.$

<u>Apparent fiducial marks</u>

If f,g are the real, measured co-ordinates of a fiducial mark on the back of the front glass and F′, G′are the apparent co-ordinates on the back of the front glass

From (I.3)

$$f = \left(\frac{P-F'}{R}\right)\Sigma\delta_i + \Sigma_1^{n-1} \delta_i \tan \beta_{(i-1)} \left\{ \frac{(F'-P)}{\sqrt{(F'-P)^2 + (G'-Q)^2}} \right\}$$

$$+ F' + \frac{\tan \beta_{n-1} (P-f) z}{\sqrt{(f-P)^2 + (g-Q)^2}}$$

$$g = \left(\frac{Q-G'}{R}\right)\Sigma\delta_i + \Sigma_1^{n-1} \delta_i \tan \beta_{(i-1)} \left\{ \frac{(G'-Q)}{\sqrt{(F'-P)^2 + (G'-Q)^2}} \right\}$$

$$+ G' + \frac{\tan \beta_{n-1} (Q-g) z}{\sqrt{(f-P)^2 + (g-Q)^2}}$$

where, except for back fiducial marks, z = 0.

Ignoring the fact that F′ + G′ are involved in a complicated fashion in tan $\beta_{(i-1)}$ etc., this gives a linear equation in F′ + G′ respectively.

$$F'\left(1-\frac{\Sigma\delta_i}{R}\right) = \sum_1^{n-1}\delta_i \tan\beta_{(i-1)} \left\{\frac{P-f}{\sqrt{(F-P)^2 + (G-Q)^2}}\right\} - \frac{P}{R}\Sigma\delta_i + f \qquad {}^{*}1 \text{ (II.5)}$$

$$G'\left(1-\frac{\Sigma\delta_i}{R}\right) = \sum_1^{n-1}\delta_i \tan\beta_{(i-1)} \left\{\frac{Q-g}{\sqrt{(F-P)^2 + (G-Q)^2}}\right\} - \frac{Q}{R}\Sigma\delta_i + g \qquad {}^{*}2 \text{ (II.6)}$$

f and g provide sufficiently good approximations to F′ and G′ to give the initial values for an iteration

$$\tan\alpha_o = \left\{\frac{\sqrt{(F'-P)^2 + (G'-Q)^2}}{R}\right\} \text{ etc.}$$

---

* For back fiducial marks there is an additional term:

*1)

$$+ \frac{(P-f)\tan\beta_{(n-1)}\, h}{\sqrt{(f-P)^2 + (g-Q)^2}}$$

*2)

$$+ \frac{(Q-g)\tan\beta_{(n-1)}\, h}{\sqrt{(f-P)^2 + (g-Q)^2}}$$

where h is the z-co-ordinate of the fiducial mark.

APPENDIX II

Approximate method of near corresponding points.

This is used to establish values of z_i for the measurements on the pivotal view when the iteration of section 5.3 breaks down. These pivotal view z_i values are then the definitive ones which are used to give values of x_i, y_i to substitute in Eq. (5). The whole procedure of section 5.3 is then gone through - but only once.

The near corresponding points method is also used to give the co-ordinates (x y z) of an apex if measurements of an apex point AA (say) are not given on two views (see section 5.2.1). The problem is:

Given $(X^I Y^I)$ defining $F_x^{\ I}$, $G_x^{\ I}$, $F_y^{\ I}$, $G_y^{\ I}$ in the reconstruction line equations

$$x = F_x^{\ I} z + G_x^{\ I}$$
$$y = F_y^{\ I} z + G_y^{\ I} \qquad \text{(II.1)}$$

to find by interpolation in all the values of $F_{xi}^{\ II}$, $G_{xi}^{\ II}$, $F_{yi}^{\ II}$, $G_{yi}^{\ II}$ on another view a set of values $F_x^{\ II}$, $G_x^{\ II}$, $F_y^{\ II}$, $G_y^{\ II}$ such that the line

$$x = F_x^{\ II} z + G_x^{\ II}$$
$$y = F_y^{\ II} z + G_y^{\ II} \qquad \text{(II.2)}$$

intersects the line Eq. (II.1) in space.

I and II do not necessarily mean views I and 2 but are symbols used to distinguish the two views chosen.

Linear interpolation is in fact used in the programme as this process is used only for an approximation.

The condition that Eqs. (II.1 or II.2) intersect is that the determinant

$$\phi(F_x^{II}, G_x^{II}, F_y^{II}, G_y^{II}) \equiv \begin{vmatrix} (F_x^{I} - F_x^{II}) & (F_y^{I} - F_y^{II}) \\ (G_x^{I} - G_x^{II}) & (G_y^{I} - G_y^{II}) \end{vmatrix} = 0. \qquad \text{(II.3)}$$

The function ϕ is evaluated for all the measurements on view II, in succession until there is a change of sign between two successive measurements i, $i+1$. As stated above a linear variation of the F's and G's between the two measurements is assumed.

Thus the required $F_x^{II} = (1-\lambda)\, F_{xi}^{II} + \lambda F_{xi+1}^{II}$ and similarly for G_x^{II}, F_y^{II}, G_y^{II} where $0 \leq \lambda \leq 1$.

Hence Eq. (II.3) may be written

$$\phi(\lambda) = 0 \qquad 0 \leq \lambda \leq 1.$$

If λ lies between λ_0 and λ_1 a better value of λ is given by

$$\lambda_2 = \frac{\lambda_0 \phi(\lambda_1) - \lambda_1 \phi(\lambda_0)}{\phi(\lambda_1) - \phi(\lambda_0)} \qquad \text{(II.4)}$$

If $\lambda_0 = 0$, and $\lambda_1 = 1$ initially then the value of λ_2 found from Eq. (II.4) takes the place of λ_0 or λ_1 for the next approximation depending on whether $\phi(\lambda)$ has the same sign as $\phi(\lambda_0)$ or $\phi(\lambda_1)$ and so on. Having found λ, F_x^{II}, G_x^{II}, F_y^{II}, G_y^{II} are easily calculated and hence from Eqs. (II.1 and II.2)

$$z = -\frac{(G_x^{I} - G_x^{II})}{(F_x^{I} - F_x^{II})} = -\frac{(G_y^{I} - G_y^{II})}{(F_y^{I} - F_y^{II})} . \qquad \text{(II.5)}$$

It has been found that one of the fractions in Eq.(II.5) can take the form $(\frac{0}{0})$. This can happen entirely due to choice of co-ordinate axis. In this case z is taken as the value of the other fraction of Eq. (II.5).

Extrapolation has been used at ends of the track on view II if necessary.

APPENDIX III

Solution of simultaneous linear equations by least squares

It happens frequently in the geometry programme that it is necessary to solve a set of linear simultaneous equations in a_j.

$$a_1 f_1(x_i) + a_2 f_2(x_i) + \ldots + a_n f_n(x_i) - y_i = 0 \;, \quad i = 1(1)N \tag{III.1}$$

where $N > n$ and the $f(x_i)$ and y_i are given. It may also be required to find the standard errors and covariances on the a_j.

The method used and the assumptions made will be given briefly without proof.

Let the $(N \times n)$ matrix of $f_j(x_i)$ be called F
$(N \times 1)$ vector of y_i be called y
$(n \times 1)$ vector of a_j be called a .

Then it is well known that the condition

$$\sum_i \left(\sum_j a_j f(x_i) - y_i \right)^2 = \text{minimum}$$

is given (on partial differentiation w.r.t the a_j in turn) by

$$\left[F'F \right] a = F'y \; . \tag{III.2}$$

This is a symmetrical set of n simultaneous linear equations in n unknowns which can be solved for the a_j .

$$a = \left[F'F \right]^{-1} \cdot F'y \tag{III.2a}$$

Further if Y_i is defined by $Y_i = \sum_j a_j f_j(x_i)$ where the a_j are found from Eq. (III.2a), and if an element of $[F'F]^{-1}$ is called c_{rs} then the variance of a_j is

$$\left(\frac{\sum (y_i - Y_i)^2}{N - n} \right) c_{jj}$$

and the covariance

$$(a_r, a_s) = \left(\frac{\sum (y_i - Y_i)^2}{N - n} \right) c_{rs} .$$

No account has been taken in this simple theory of any original known measurement errors on the x_i.

APPENDIX IV

Results of error analysis

As stated in section 5.4.4 the same artificial event was punched on tape 57 times with different "measurement errors" every time; the 57 events thus created were computed by the geometry programme and the results analysed by an "analysis programme". Both of these programmes (for the artificial creation of the events and for the analysis) are by R. Böck.

A sample result from the geometry programme is:

```
EVENT 380091      REF. 1000      1      3

APEX  E -0.4972  0.0038  -4.7663    0.0039   -2.4325      0.0311

TRACK  E1    8      1
         66.50    -0.00769     1.20790     0.93487     0.35498
   5.5446, -1      2.0680, -3     7.6743,  -4
   6.1234, -5      4.1210, -4    -3.2564,  -8
     -0.001          0.001          0.000          0.000       0.16117
  -0.49718     0.00000   -4.76711     0.00216    -2.44211      0.01515

TRACK  E2      4      1
         53.23     0.84566     1.29861   0.96319       0.26884
   1.7029, -1      1.2023, -3      3.9013, -4
   3.5045, -5     -6.2992, -5     -1.5836, -7
     0.000           0.000          0.000          0.000      -0.25006
  -0.48718     0.00000    -4.76816     0.00117    -2.44493      0.00809

TRACK  E3      6      2
         8.40     -2.19818      1.00998     0.84682       0.53187
   5.1461, -1      1.9311, -2      5.1617, -3
   2.5989, -4     -2.5722, -3     -7.2193, -6
     0.003          -0.003          0.001          0.000      -0.16566
  -0.49718     0.00000    -4.76874     0.00184   -2.43902      0.01492
```

The "true" results are given in the first column of the next table which shows the error analysis.

ERROR ANALYSIS - Using the results of 57 artificial events

	Actual	Mean	Error of Mean*	Mean of Error
A	- .4971	-0.496726	0.001946	0.002080
B	-4.7692	-4.769458	0.002184	0.002168
C	-2.4214	-2.421795	0.016230	0.017080
ρ	66.58	66.639123	0.447850	0.635228
tan α	- .12003	-0.119721	0.002782	0.002387
β	1.20778	1.207844	0.00625	0.000882
ρ	53.16	53.159824	0.229159	0.250010
tan α	.84490	0.844390	0.002148	0.001770
β	1.29873	1.298783	0.000582	0.000609
ρ	8.64	8.676491	0.615192	0.568210
tan α	-2.20041	-2.198344	0.020870	0.020749
β	1.00625	1.006148	0.005555	0.005531

*Square root of variance of the computed quantities about the mean.

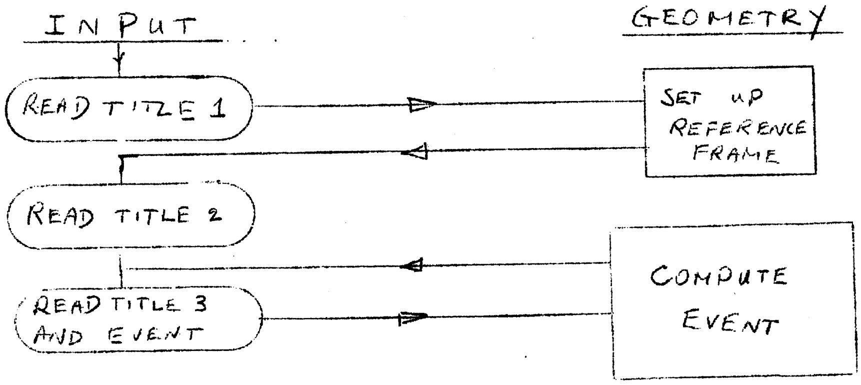


Fig. (i) Connections between input and geometry programmes.

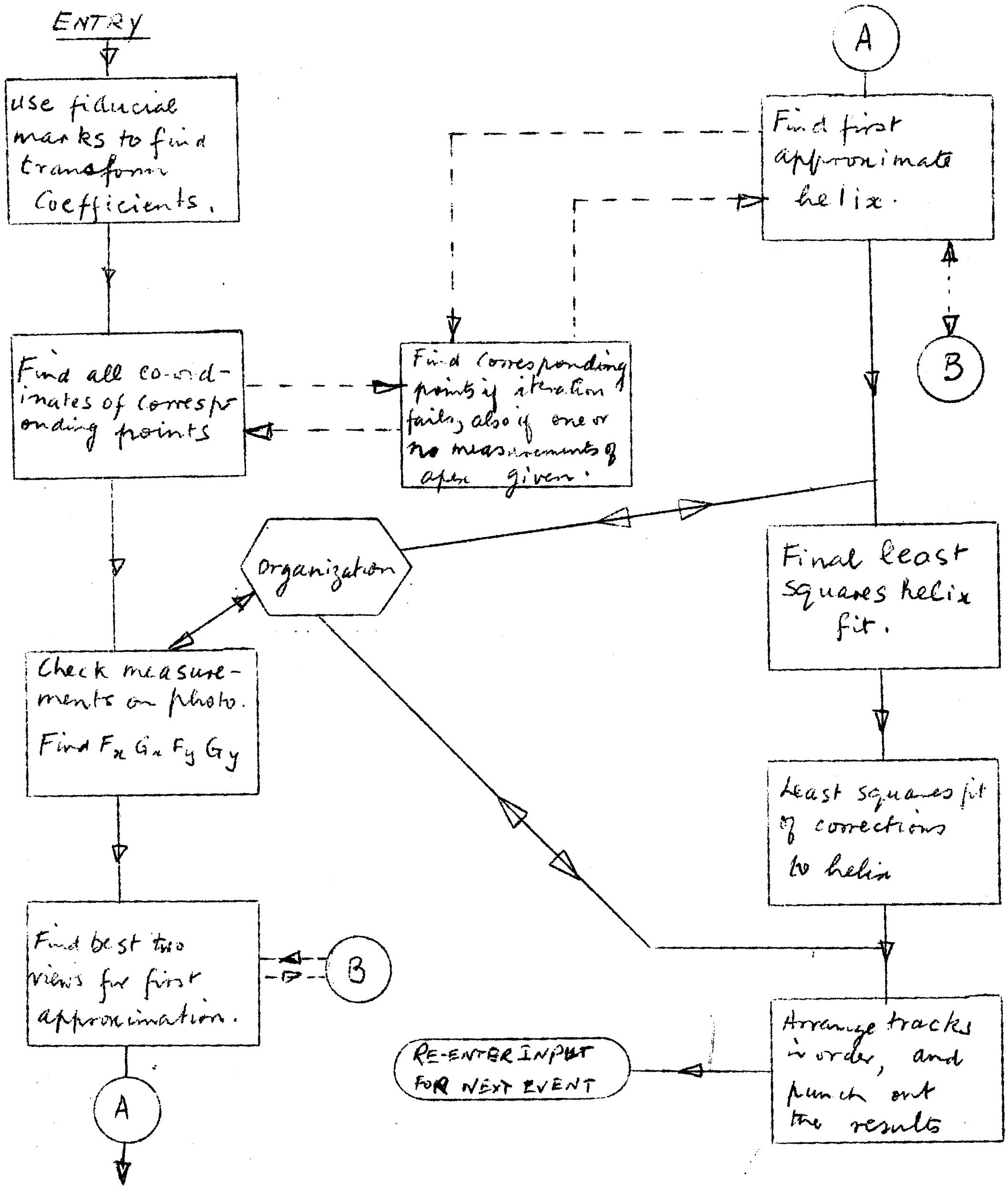
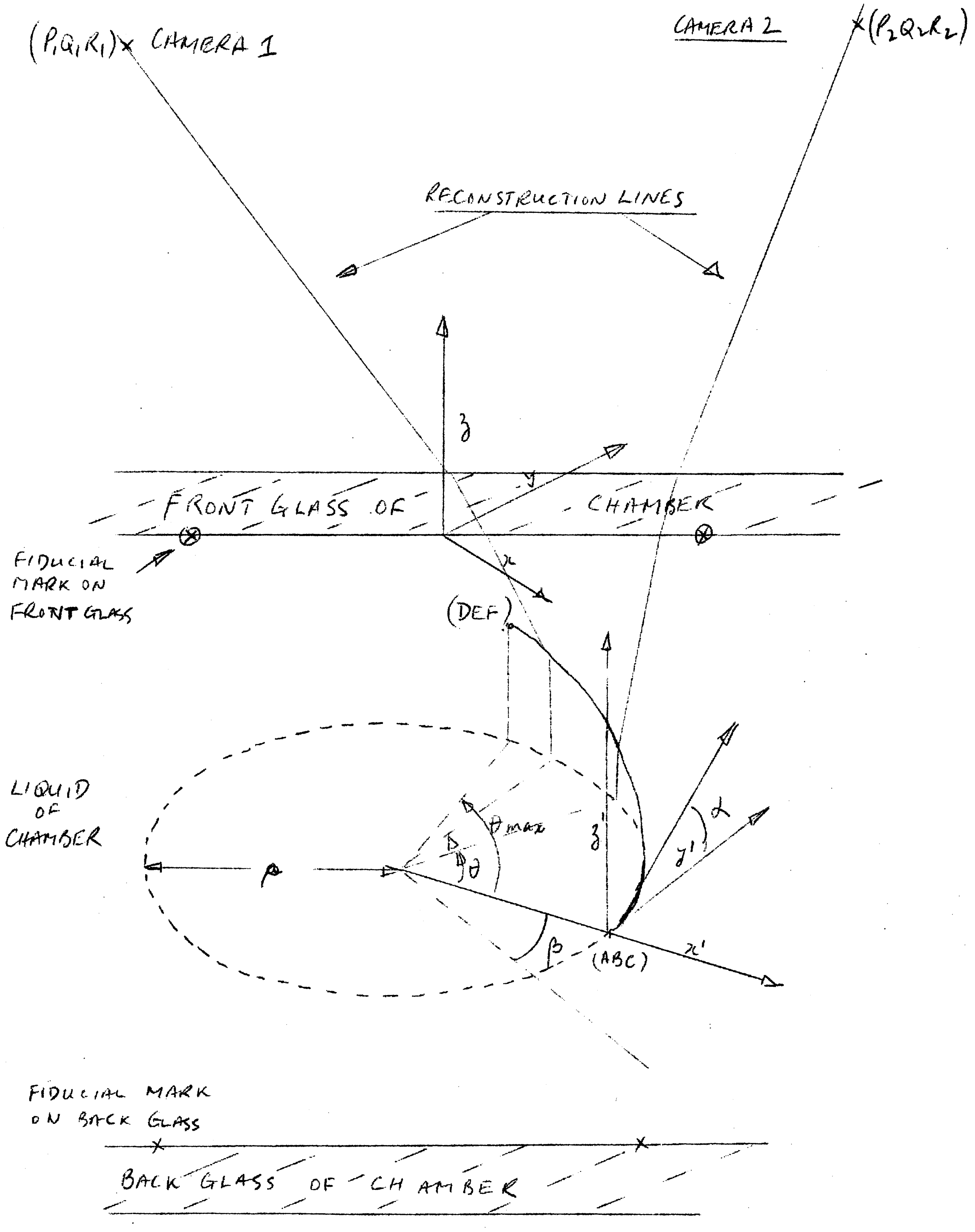


Fig. (II) General flow diagram of geometry programme for each event.



Fiv. (iv) Illustration of co-ordinate system. One track is shown and a typical reconstruction line from each of two views intersecting it.